THE CHINESE UNIVERSITY OF HONG KONG
Department of Mathematics
MATH3070 (Second Term, 2014–2015)
Introduction to Topology
Exercise 7 Product and Quotient

Remarks

Many of these exercises are adopted from the textbooks (Davis or Munkres). You are suggested to work more from the textbooks or other relevant books.

1. Show that the relative topology (induced topology) is "transitive" in some sense. That is for $A \subset B \subset (X, \mathcal{T})$, the topology of A induced indirectly from B is the same as the one directly induced from X.

2. Let $A \subset (X, \mathcal{T})$ be given the induced topology $\mathcal{T}|_A$ and $B \subset A$. Guess and prove the relation between $\mathrm{Int}_A(B)$ and $\mathrm{Int}_X(B)$ which are the interior wrt to $\mathcal{T}|_A$ and $\mathcal{T}$. Do the similar thing for closures.

3. Let $A \subset (X, \mathcal{T})$ be given a topology $\mathcal{T}_A$. Formulate a condition for $\mathcal{T}_A$ being the induced topology in terms of the inclusion mapping $\iota\colon A \hookrightarrow X$.

4. Let $Y \subset (X, \mathcal{T})$ be a closed set which is given the induced topology. If $A \subset Y$ is closed in $(Y, \mathcal{T}|_Y)$, show that A is also closed in $(X, \mathcal{T})$.

5. Let $X \times X$ be given the product topology of $(X, \mathcal{T})$. Show that $D = \{ (x, x) : x \in X \}$ as a subspace of $X \times X$ is homeomorphic to X.

6. Let Y be a subspace of $(X, \mathcal{T})$, i.e., with the induced topology and $f\colon X \to Z$ be continuous. Is the restriction $f|_Y\colon Y \to Z$ continuous?

7. Show that $(X \times Y) \times Z$ is homeomorphic to $X \times (Y \times Z)$ wrt product topologies.

8. Let $X_1 \times X_2$ be given the product topology. Show that the mappings $\pi_j\colon X_1 \times X_2 \to X_j$, $j = 1, 2$, are open and continuous.

   Moreover, let $\mathcal{T}^*$ be a topology on $X_1 \times X_2$ such that both mappings
   $$\pi_j\colon (X_1 \times X_2, \mathcal{T}^*) \to (X_j, \mathcal{T}_j), \quad j = 1, 2,$$
   are continuous. What is the relation between $\mathcal{T}^*$ and the product topology?

9. Given any topological space Y and product space $X_1 \times X_2$, a mapping $f\colon Y \to X_1 \times X_2$ is continuous if and only if $\pi_j \circ f$, $j = 1, 2$, are continuous.

   If $\mathcal{T}^*$ is a topology on $X_1 \times X_2$ with the same property, then $\mathcal{T}^*$ is the product topology.

10. Let $X = \{ (x, 0) : x \in \mathbb{R} \} \cup \{ (x, 1) : x \in \mathbb{R} \} \subset \mathbb{R}^2$, i.e., $X = \mathbb{R} \coprod \mathbb{R}$. Define an equivalence relation on X by identifying $(0, 0)$ and $(0, 1)$. Rigorously, this means $(s_1, t_1) \sim (s_2, t_2)$ iff $(s_1, t_1) = (s_2, t_2)$ or $(s_1, t_1) = (0, 0)$ while $(s_2, t_2) = (0, 1)$ or vice versa. Show that $X/\!\sim$ is homeomorphic to the two axes in $\mathbb{R}^2$.

11. Let $X = \{ (s, t) \in \mathbb{R}^2 : 0 \neq t \in \mathbb{Z} \}$ and $Y = \{ (s, t) \in \mathbb{R}^2 : 1/t \in \mathbb{Z} \}$ be given the standard induced topology. Define an equivalence relation on both X and Y by $(s_1, t_1) \sim (s_2, t_2)$ iff $(s_1, t_1) = (s_2, t_2)$ or $s_1 = s_2 = 0$. That is points on the y-axis are identified to one point. Is it true that $X/\!\sim$ and $Y/\!\sim$ are homeomorphic?

12. Define an equivalence relation on $\mathbb{R}$ by identifying n with $1/n$ for all $n \in \mathbb{Z}$.

    (a) Sketch a picture to represent the space $\mathbb{R}/\!\sim$.

    (b) Find a sequence $x_n \in \mathbb{R}$ such that $[x_n] \in \mathbb{R}/\!\sim$ converges but x_n does not.

    (c) Can a sequence $x_n \in \mathbb{R}$ converge but $[x_n] \in \mathbb{R}/\!\sim$ does not?

13. Let $X/\!\sim$ be a quotient space obtained from X and $Y \subset X$.

    (a) Show that there is a natural way to induce an equivalence relation on Y; and thus a quotient space $Y/\!\sim$.

    (b) Let $Y^* = \{ [x] \in (X/\!\sim) : [x] \cap Y \neq \emptyset \}$ be given the topology induced from $X/\!\sim$. Is Y^* homeomorphic to $Y/\!\sim$?

14. Let $X = \mathbb{R}/\!\sim$ where $s \sim t$ if $s, t \in \mathbb{Z}$ and $Y = \bigcup_{n=1}^{\infty} \{ z \in \mathbb{C} : \left| z - \frac{1}{n} \right| = \frac{1}{n} \}$ with induced topology of $\mathbb{C} = \mathbb{R}^2$. Is X homeomorphic to Y?

15. Let $A \subset X$. What can you say about $\mathrm{Int}(A)/\!\sim$ and $\mathrm{Int}(A/\!\sim)$; $\mathrm{Cl}(A)/\!\sim$ and $\mathrm{Cl}(A/\!\sim)$?

16. Let $\mathcal{T}_q$ be the quotient topology on $X/\!\sim$ and $\mathcal{T}'$ be any topology. Show that if the quotient map $q\colon (X, \mathcal{T}) \to (X/\!\sim, \mathcal{T}')$ is continuous, then $\mathcal{T}' \subset \mathcal{T}_q$.

17. Let Z be any topological space. A mapping $f\colon (X/\!\sim, \mathcal{T}_q) \to Z$ is continuous if and only if $f \circ q\colon (X, \mathcal{T}) \to Z$ is continuous.

    If $\mathcal{T}'$ is a topology on $X/\!\sim$ satisfying the same property, then $\mathcal{T}' = \mathcal{T}_q$.

18. Let (X_n, d_n), $n \in \mathbb{N}$, be a countable family of metric spaces; $X = \prod_{n=1}^{\infty} X_n$ be the product space of the metric topologies induced by d_n. Define a metric d on X in this way, for $x = (x_n), y = (y_n) \in X$,

$$d(x, y) = \sum_{n=1}^{\infty} \frac{1}{2^n} \frac{d_n(x_n, y_n)}{1 + d_n(x_n, y_n)} .$$

    Show that d is a metric on X and the topology it induces is exactly the product topology.